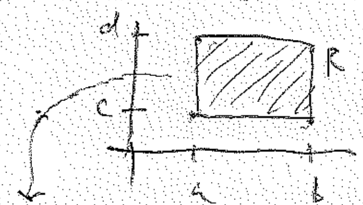
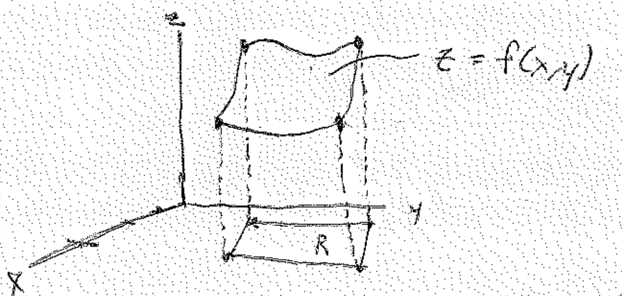
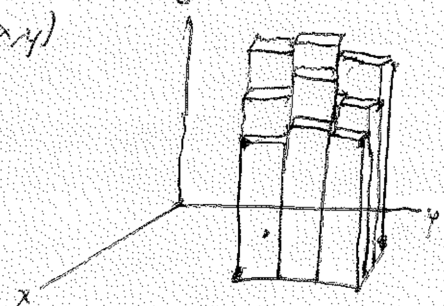
Ch. 6 Double and triple integrals

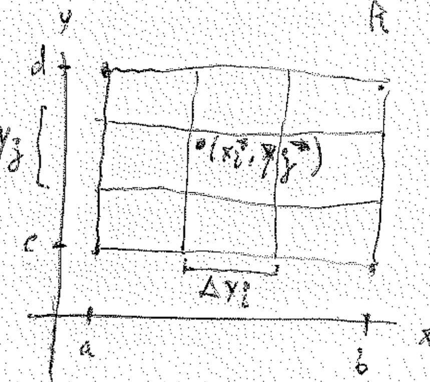
Sec 6.1 Double integrals over rectangles.

- **Def** A rectangle is $R = [a,b] \times [c,d] = \{(x,y) : a \le x \le b,\ c \le y \le d\}$









Volume of solid above R and below $z = f(x,y) \approx$

$$R_n = \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i^*, y_j^*) \underbrace{\Delta x_i \, \Delta y_j}_{\Delta A_{ij}}$$

$$\lim_{n \to \infty} R_n = \int_c^d \int_a^b f(x,y)\, dx\, dy = \iint_R f \, dA.$$

If $f \ge 0$, $\iint_R f \, dA =$ volume under $z = f(x,y)$.

Example $\displaystyle \iint_{[0,1]\times[2,4]} e^{2x+3y}\, dA = \int_2^4 \int_0^1 e^{2x+3y}\, dx\, dy = \int_2^4 \left[\tfrac{1}{2} e^{2x+3y}\right]_0^1 dy$

$$= \frac{1}{2}\int_2^4 e^{2+3y} - e^{3y}\, dy$$

$$= \frac{1}{2}\left[\frac{1}{3}e^{2+3y} - \frac{1}{3}e^{3y}\right]_2^4$$

$$= \frac{1}{6}\left(e^{14} - e^{12} - e^{8} + e^{6}\right).$$

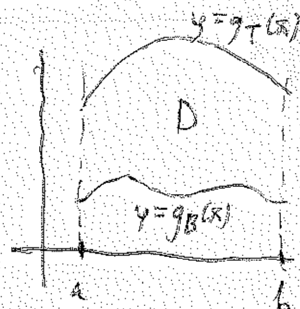
- **Fubini's Theorem**: If f is continuous ~~and bounded~~ on a rectangle R, then $\iint_R f(x,y)\,dA = \int_c^d\int_a^b f\,dx\,dy = \int_a^b\int_c^d f\,dy\,dx$.

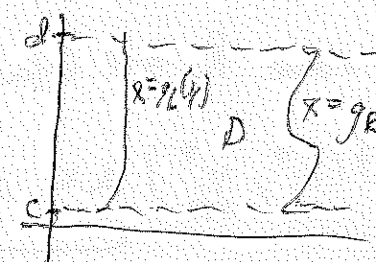
Example Evaluate $\iint_R y\sin(xy)\,dA$ where $R=[1,2]\times[0,\pi]$

Soln $\int_1^2\int_0^\pi y\sin(xy)\,dy\,dx$ requires integration by parts.

But
$$\int_0^\pi\int_1^2 y\sin(xy)\,dx\,dy = \int_0^\pi \Big[-\cos(xy)\Big]_1^2\,dy$$
$$= \int_0^\pi -\cos 2y + \cos y\,dy$$
$$= \Big[-\tfrac{1}{2}\sin 2y + \sin y\Big]_0^\pi = 0.$$

Sec 6.2 Double integrals over general regions





- **Def** Type 1 region: $D=\{(x,y): a\le x\le b,\ g_B(x)\le y\le g_T(x)\}$

Def Type 2 region: $D=\{(x,y): g_L(y)\le x\le g_R(y),\ c\le y\le d\}$

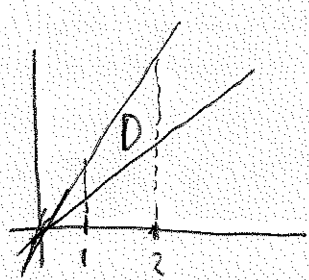
- How to compute: if $f: D\subseteq\mathbb{R}^2\to\mathbb{R}$ is continuous, then

① $\iint_D f\,dA = \int_a^b\int_{g_B(x)}^{g_T(x)} f\,dy\,dx$ if D is type 1

② $\iint_D f\,dA = \int_c^d\int_{g_L(y)}^{g_R(y)} f\,dx\,dy$ if D is type 2.

Example Compute $\iint_D e^{2x+y}\, dA$ where D is the region bounded by $y=2x$, $y=x$, $x=1$, and $x=2$.

Soln



$D = \{(x,y) : 1 \le x \le 2,\ x \le y \le 2x\}$ is type 1.

$$\iint_D e^{2x+y}\, dA = \int_1^2 \int_x^{2x} e^{2x+y}\, dy\, dx = \int_1^2 \left[e^{2x+y} \right]_x^{2x} dx$$

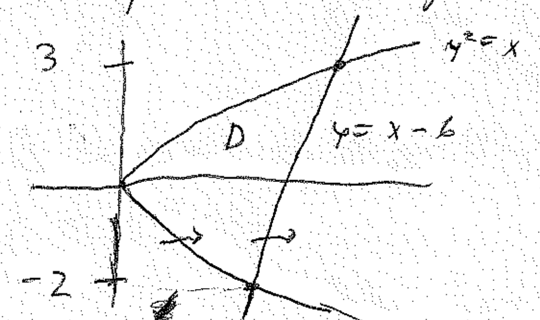
$$= \int_1^2 e^{4x} - e^{3x}\, dx$$

$$= \left[\frac{e^{4x}}{4} - \frac{e^{3x}}{3} \right]_1^2$$

$$= \frac{e^8}{4} - \frac{e^6}{3} - \frac{e^4}{4} + \frac{e^3}{3}.$$

Example $\iint_D 2y\, dA$, where D is the region bounded by $y = x-6$ and $y^2 = x$.

Soln



Intersection points:

$y^2 = y+6$

$y^2 - y - 6 = (y+2)(y-3) = 0$

$y = -2, 3$

D is both type 1 and 2, but more easily viewed as type 2

$D = \{(x,y) : y^2 \le x \le y+6,\ -2 \le y \le 3\}$

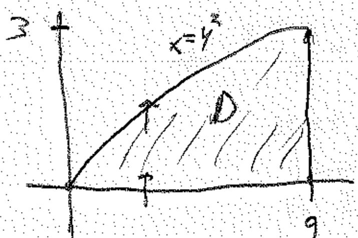
$$\iint_D 2y \, dA = \int_{-2}^{3}\int_{y^2}^{y+6} 2y \, dx\, dy = 2\int_{-2}^{3}\Big[xy\Big]_{y^2}^{y+6} dy$$

$$= 2\int_{-2}^{3} (y+6)y - y^3 \, dy$$

$$= 2\int_{-2}^{3} y^2 + 6y - y^3 \, dy = \dots = \frac{125}{6}.$$

<u>Example</u> Consider $\int_0^3 \int_{y^2}^{9} dx\, dy$.

So $D = \{(x,y) : y^2 \le x \le 9, \ 0 \le y \le 3\}$ is type 2.



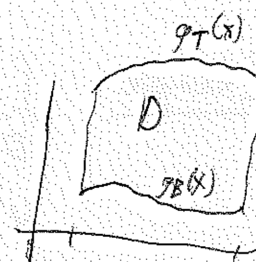
Note that D is also type 1:

$D = \{(x,y) : 0 \le x \le 9, \ 0 \le y \le \sqrt{x}\}$

Thus $\int_0^3 \int_{y^2}^{9} f \, dx\, dy = \int_0^9 \int_0^{\sqrt{x}} f \, dy\, dx$.

• <u>Applications</u>:

• Area of $D = \iint_D dA$.

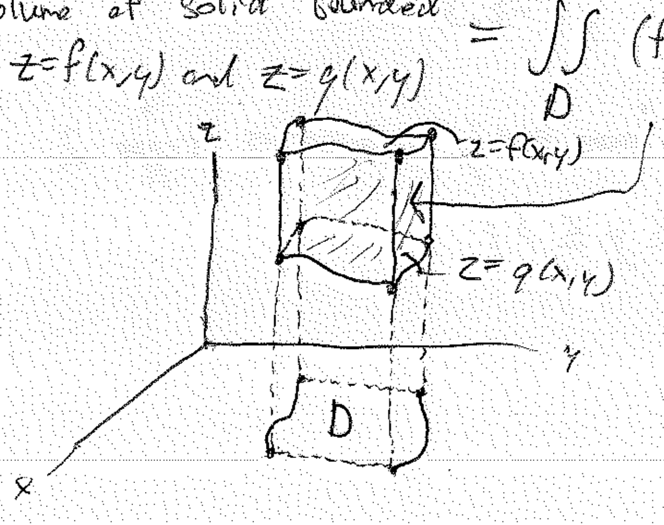


Why? If D is type 1, $\iint_D dA = \int_a^b \int_{g_B(x)}^{g_T(x)} dy\, dx$

$$= \int_a^b g_T(x) - g_B(x) \, dx = \text{area of } D$$

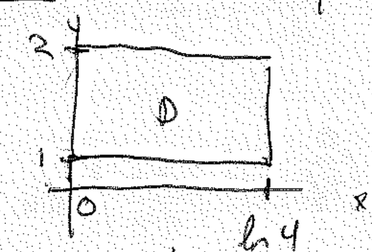
(similarly for type 2)

• Volume of solid bounded by $z = f(x,y)$ and $z = g(x,y)$ $= \iint_D (f-g) \, dA$.



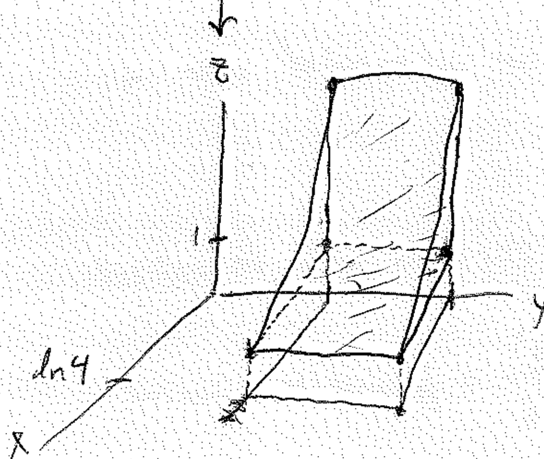
Example Volume of solid bounded above by $z = 4e^{-x}$, below by $z = 1$, and from the sides by $y=1, y=2$, and $x=0$.

Soln D in xy-plane:



Intersection point:

$4e^{-x} = 1 \longrightarrow e^{-x} = \frac{1}{4} \rightarrow x = -\ln\frac{1}{4} = \ln 4.$



$$\text{Volume} = \iint_D (4e^{-x} - 1)\, dA$$

$$= \int_1^2 \int_0^{\ln 4} 4e^{-x} - 1 \, dx\, dy$$

$$= \int_1^2 \left[-4e^{-x} - x \right]_0^{\ln 4} dy$$

$$= \int_1^2 3 - \ln 4 \, dy = 3 - \ln 4.$$

Sec 6.3 Double integral techniques and examples.

- If $f(x,y) = g(x)h(y)$ and $R = [a,b] \times [c,d]$, then

$$\iint_R f\, dA = \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right).$$

Example $\int_0^1\int_0^1 e^{x+y}\, dx\, dy = \int_0^1\int_0^1 e^x e^y\, dx\, dy$

$$= \left(\int_0^1 e^x\, dx\right)\left(\int_0^1 e^y\, dy\right)$$

$$= (e-1)(e-1).$$

Sec 6.4 Change of variables

- In one variable (u-substitution)

$$f(x) = -2x$$

$$\int_{x=0}^{2} -2x \, dx. \quad \text{Let } u = -2x, \text{ so } -\frac{1}{2}du = dx \text{ and } \left|\frac{dx}{du}\right| = \frac{1}{2}, \; f(x(u)) = u$$

$$\rightarrow x(u) = -\frac{1}{2}u$$

and $$\int_{x=0}^{2} -2x \, dx = \int_{u=0}^{-4} u \cdot \left(-\frac{1}{2}du\right) = \int_{u=-4}^{0} u \cdot \frac{1}{2} du = \int_{u=-4}^{0} f(x(u)) \left|\frac{dx}{du}\right| du.$$

$\left|\frac{dx}{du}\right| = \frac{1}{2}$ kept track of the fact that we integrated over a triangle that was twice as big.

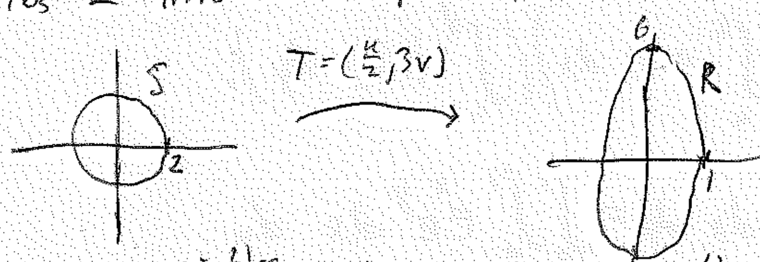
- The equations that define a change of variable will be called a <u>transformation</u>.

<u>Example</u> Let R be the ellipse $x^2 + \frac{y^2}{36} = 1$, and its interior.

Consider the transformation $x = \frac{u}{2}$, $y = 3v$

Then $\left(\frac{u}{2}\right)^2 + \frac{(3v)^2}{36} = \frac{u^2}{4} + \frac{v^2}{4} = 1 \rightarrow u^2 + v^2 = 4$

Thus $T(u,v) = (x(u,v), y(u,v)) = \left(\frac{u}{2}, 3v\right)$ transforms the disk S of radius 2 into the ellipse R, $T: S \rightarrow R$



If we change variables and integrate over the disk S instead of R, we need to keep track of the change in area.

<u>Def</u> The <u>Jacobian</u> of $T(u,v) = (x(u,v), y(u,v))$ is

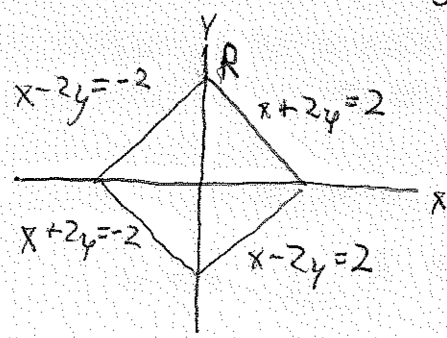
$$\det DT = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

- **Theorem** (<u>Change of variables</u>): Let S, R be type 1 or 2 regions, and suppose T is a C^1 transformation, with nonzero Jacobian, that maps S onto R. Suppose also that T is 1-1, except possibly on the boundary of S. If f is a function continuous on R, then

$$\boxed{\iint_R f(x,y)\,dA = \iint_S \overbrace{f(x(u,v),y(u,v))}^{=f(T(u,v))} \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\text{absolute value}} du\,dv}$$

- <u>Note</u> $dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$.

<u>Example</u> Consider $\iint_R (3x+6y)^2\,dA$, where R is:



Then $\iint_R (3x+6y)\,dA = \int_{-2}^{0}\int_{\frac{-2-x}{2}}^{\frac{x+2}{2}} (3x+6y)^2\,dy\,dx + \int_0^2 \int_{\frac{x-2}{2}}^{\frac{2-x}{2}} (3x+6y)^2\,dy\,dx$

We'll instead apply a change of variables to integrate over a simpler region.

Let $u = x+2y$ and $v = x-2y$ and notice that $x = \frac{1}{2}(u+v)$, $y = \frac{1}{4}(u-v)$, so

$T: S \to R$, $T(u,v) = \left(\frac{1}{2}(u+v), \frac{1}{4}(u-v)\right)$ transforms the square $S = [-2,2]\times[-2,2]$ into R.

Why? $x+2y=2 \leftrightarrow u=2$ $\quad x+2y=-2 \leftrightarrow u=-2$

$x-2y=2 \leftrightarrow v=2$ $\quad x-2y=-2 \leftrightarrow v=-2$

Our new integrand is

$$(3x+6y)^2 = \left(3\cdot\frac{u+v}{2} + 6\cdot\frac{u-v}{4}\right)^2 = \left(\frac{6u+\cancel{6v}+6u-\cancel{6v}}{4}\right)^2 = 9u^2$$

Next, $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \text{abs}\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \text{abs}\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{-1}{4} \end{vmatrix} = \left|\frac{-1}{4}\right| = \frac{1}{4}.$

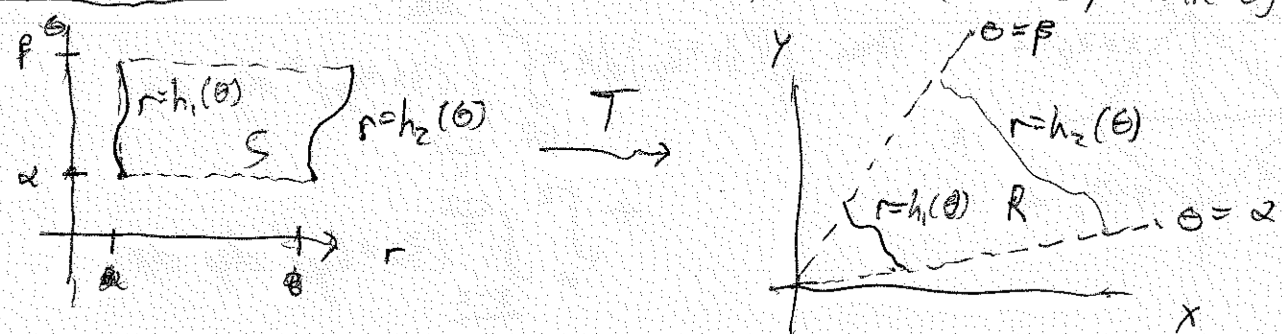
(note: area of $R = 4$, area of $S = 16$).

Thus

$$\iint_R (3x+6y)^2\, dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

$$= \int_{-2}^{2}\int_{-2}^{2} 9u^2 \cdot \frac{1}{4}\, du\, dv = \underset{\text{easier}}{\text{much}} = 48.$$

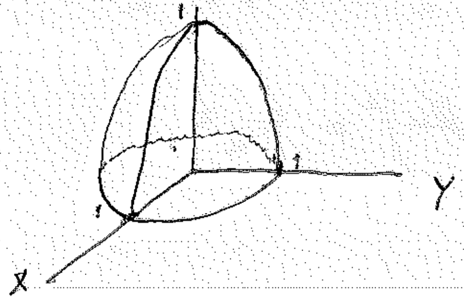
- <u>Polar coordinates</u>: $T(r,\theta) = (x(r,\theta), y(r,\theta)) = (r\cos\theta, r\sin\theta)$



$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \text{abs}\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \text{abs}\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = |r| = r$$

So
$$\boxed{\begin{array}{l} dA = r\, dr\, d\theta \\[1em] \displaystyle\iint_R f(x,y)\, dA = \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta. \end{array}}$$

<u>Example</u> Find the volume of the solid bounded by $z=0$ and $z = 1-x^2-y^2$.



Approach 1:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1-x^2-y^2\, dy\, dx.$$

Approach 2:

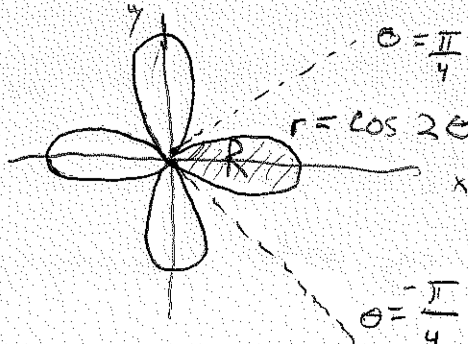
$$\int_0^{2\pi}\int_0^1 \left[1-(r\cos\theta)^2-(r\sin\theta)^2\right] r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}-\frac{r^3}{3}\right]_0^1 d\theta$$

$$= \int_0^{2\pi}\frac{1}{4}\,d\theta$$

$$= \boxed{\frac{\pi}{2}}.$$

<u>Example</u> Find the area of R:



<u>Soln</u> area of $R = \iint_R dA$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\cos 2\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{4}}\left[\frac{1}{2}r^2\right]_0^{\cos 2\theta} d\theta$$

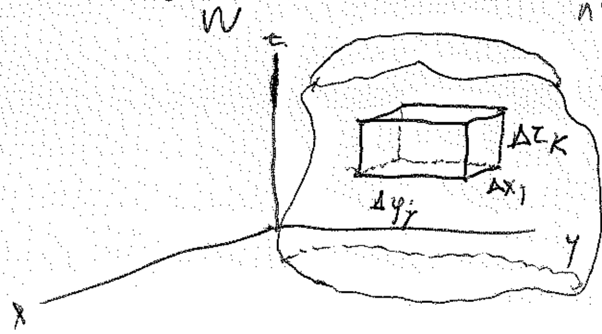
$$= \int_0^{\frac{\pi}{4}}\cos^2 2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} 1+\cos 4\theta\,d\theta = \dots = \boxed{\frac{\pi}{8}}.$$

<u>Sec 6.5</u> Triple integrals

- <u>Def</u> Assume f is bounded on a closed bounded solid W in $\mathbb{R}^3$.

Then $$\iiint_W f(x,y,z)\,dV = \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f(x_i^*,y_j^*,z_k^*)\underbrace{\Delta x_i\,\Delta y_j\,\Delta z_k}_{\Delta V_{ijk}}$$



Example $B = [0,1] \times [0,2] \times [0,3]$

$$\iiint_B xyz^2 \, dV = \int_0^3 \int_0^2 \int_0^1 xyz^2 \, dx \, dy \, dz$$

$$= \int_0^3 \int_0^2 \left[\frac{x^2 y z^2}{2} \right]_0^1 dy \, dz$$

$$= \int_0^3 \int_0^2 \frac{yz^2}{2} \, dy \, dz$$

$$= \int_0^3 \left[\frac{y^2 z^2}{4} \right]_0^2 dz$$

$$= \int_0^3 z^2 \, dz = \left[\frac{z^3}{3} \right]_0^3 = \boxed{9.}$$

- An analog of Fubini's theorem holds: if f is continuous ~~and bounded~~ on a "box" $B = [a,b] \times [c,d] \times [e,f]$, then $\iiint_B f(x,y,z) \, dV$ can be computed in any order (6 ways).

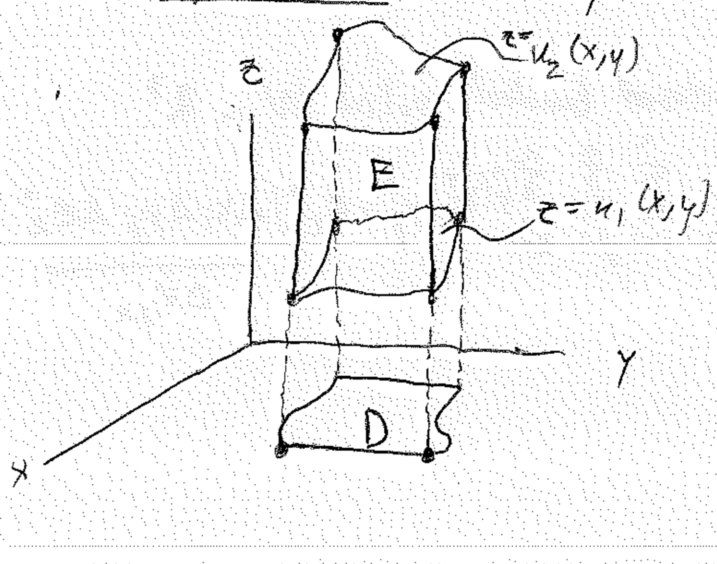
- Type 1, 2, and 3 regions in 3D

Let D be a type 1 or 2 region in 2D.

- Def Type 1 (3D) $E = \{(x,y,z) : (x,y) \in D, \; u_1(x,y) \leq z \leq u_2(x,y)\}$

  Type 2 (3D) Switch x and z.

  Type 3 (3D) Switch y and z.



· <u>How to compute</u> !

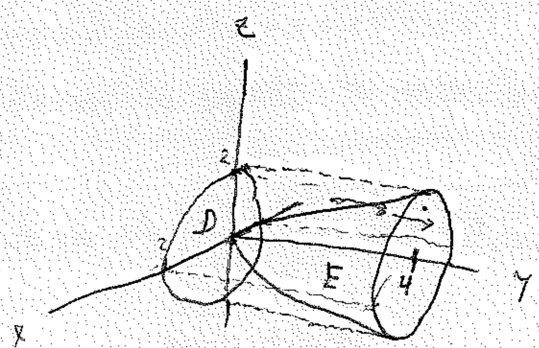
Type 1: $\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$

Type 2 and 3 triple integrals are defined analogously.

<u>Example</u> $\iiint_E \sqrt{x^2+z^2}\,dV$, where E is the region bounded by

$y = x^2 + z^2$ and $y = 4$.

<u>Soln</u>



Project onto xz-plane:

D = disk of radius 2, $x^2+y^2 \le 4$

$D = \{(x,z) : -2 \le x \le 2, -\sqrt{4-x^2} \le z \le \sqrt{4-x^2}\}$

$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy \right) dA$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\,dz\,dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-x^2-z^2)\sqrt{x^2+z^2}\,dz\,dx$$

$$= \int_0^{2\pi} \int_0^2 (4-r^2)\cdot r \cdot r\,dr\,d\theta \qquad \begin{pmatrix} x = r\cos\theta \\ z = r\sin\theta \\ dz\,dx = r\,dr\,d\theta \end{pmatrix}$$

$$= \int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5} \right]_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{32}{3} - \frac{32}{5}\,d\theta = \boxed{2\pi\left(\frac{32}{3} - \frac{32}{5}\right)}.$$

- <u>Recall</u> $\iint_D dA$ = area of D.

- Similarly, $\iiint_E dV$ = Volume of E

Why? If E is type 1, for example, then

$$\iiint_E dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} dz \right) dA = \iint_D \left[u_2(x,y) - u_1(x,y)\right] dA$$

- <u>Change of variables</u>: Let $T: S \to R$ be $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$.

Under hypotheses similar to the double integral version of change of variables,

$$\boxed{\iiint_R f(x,y,z)\, dV = \iiint_S \overbrace{f(x(u,v,w), y(u,v,w), z(u,v,w))}^{f(T(u,v,w))} \underbrace{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|}_{\text{absolute value}} du\, dv\, dw}$$

<u>Note</u> $dV = \left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw$

<u>Example</u> Volume of ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

<u>Soln</u> Let $x = au$, $y = bv$, $z = cw$.

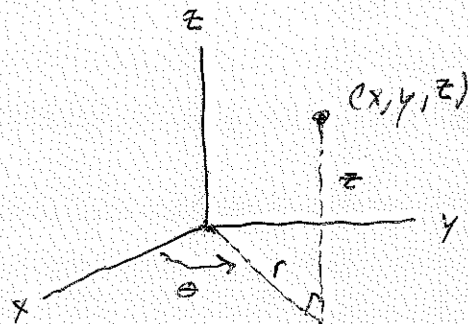
Then $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ becomes $u^2 + v^2 + w^2 = 1$, and

$T(u,v,w) = (au, bv, cw)$ maps the unit ball S to the ellipsoid and interior, R.

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \text{abs}\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = |abc| = abc,$$

So volume $= \iiint_R dV = abc \iiint_S du\, dv\, dw = abc \cdot \frac{4}{3}\pi 1^3 = \boxed{\frac{4\pi abc}{3}}$.

- Cylindrical coordinates



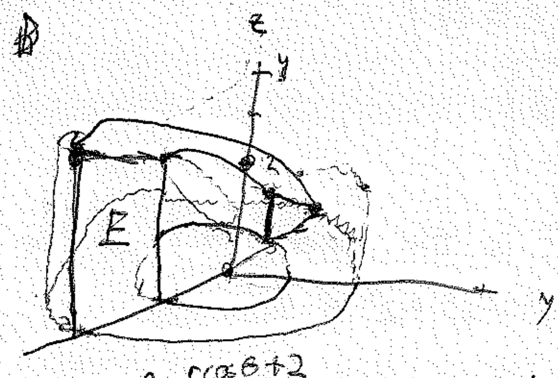
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \text{abs}\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix} = |r| = r,$$

so $\boxed{dV = r\,dz\,dr\,d\theta}$

Example $\iiint_E y\,dV$ where E is the region below the plane $z = x+2$, above the xy-plane, and between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$.

Soln



Bounds: $0 \le z \le x+2 \longrightarrow 0 \le z \le r\cos\theta + 2$
$0 \le \theta \le 2\pi$
$1 \le r \le 2$

$$\iiint_E y\,dV = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} (r\sin\theta)\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_1^2 r^2\sin\theta(r\cos\theta+2)\,dr\,d\theta =$$

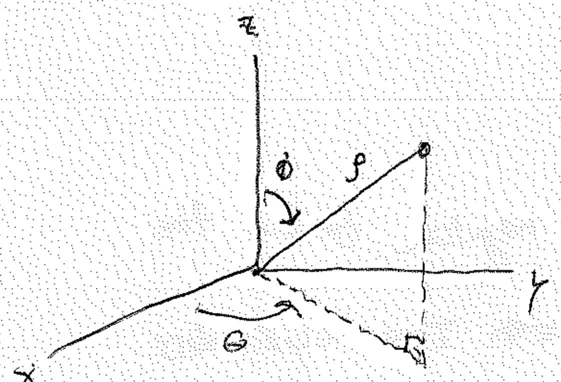
$$= \int_0^{2\pi}\int_1^2 \frac{1}{2}r^3 \sin 2\theta + 2r^2 \sin\theta \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{8}r^4 \sin 2\theta + \frac{2}{3}r^3 \sin\theta\right]_1^2 d\theta$$

$$= \int_0^{2\pi} \frac{15}{8}\sin 2\theta + \frac{14}{3}\sin\theta \, d\theta$$

$$= \left[\frac{-15}{16}\cos 2\theta - \frac{14}{3}\cos\theta\right]_0^{2\pi} = \boxed{0}.$$

- Spherical coordinates



$x = \rho \sin\phi \cos\theta$ $\quad x^2+y^2+z^2 = \rho^2$

$y = \rho \sin\phi \sin\theta$ $\quad \rho \geq 0$

$z = \rho\cos\phi$ $\quad 0 \leq \phi \leq \pi$

$\quad 0 \leq \theta \leq 2\pi$

A computation shows that $\left|\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2 \sin\phi$

so $\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}$

Example $\iiint_E 16z \, dV$, where E is the upper half of the unit ball $x^2+y^2+z^2 \leq 1$.

Soln Bounds: $0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \frac{\pi}{2}$.

$$\iiint_E 16z \, dV = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 (16\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 8\rho^3 \sin(2\phi) \, d\rho \, d\theta \, d\phi \quad = \quad \longrightarrow$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 2 \sin(2\phi) \, d\theta \, d\phi$$

$$= \int_0^{\pi/2} 4\pi \sin(2\phi) \, d\phi$$

$$= \left[-2\pi \cos(2\phi) \right]_0^{\frac{\pi}{2}} = \boxed{4\pi}.$$